

Induced subgraphs with large degrees at end-vertices for hamiltonicity of claw-free graphs*

Roman Čada^a, Binlong Li^{a,b†}, Bo Ning^{b‡} and Shenggui Zhang^{b§}

^a Department of Mathematics, NTIS - New Technologies for the Information Society,
University of West Bohemia, 30614 Pilsen, Czech Republic

^b Department of Applied Mathematics, School of Science,
Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China

Abstract

A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. Matthews and Sumner proved that a 2-connected claw-free graph G is hamiltonian if every vertex of it has degree at least $(|V(G)| - 2)/3$. At the workshop C&C (Novy Smokovec, 1993), Broersma conjectured the degree condition of this result can be restricted only to end-vertices of induced copies of N (the graph obtained from a triangle by adding three disjoint pendant edges). Fujisawa and Yamashita showed that the degree condition of Matthews and Sumner can be restricted only to end-vertices of induced copies of Z_1 (the graph obtained from a triangle by adding one pendant edge). Our main result in this paper is a characterization of all graphs H such that a 2-connected claw-free graph G is hamiltonian if each end-vertex of every induced copy of H in G has degree at least $|V(G)|/3 + 1$. This gives an affirmative solution of the conjecture of Broersma up to an additive constant.

Keywords: induced subgraph; large degree; end-vertex; claw-free graph; hamiltonian graph

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G , we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H , respectively. We call $d_H(v)$ the *degree* of v in H . For $x, y \in V(G)$, an (x, y) -*path* is a path connecting x and y . If $x, y \in V(H)$, the *distance* between x and y in H , denoted $d_H(x, y)$, is the length of a shortest (x, y) -path in H . When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G(x, y)$ by $N(v)$, $d(v)$ and $d(x, y)$, respectively.

Let G be a graph and G' a subgraph of G . If G' contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph *induced by* $V(G')$).

*Supported by NSFC (11271300), the Doctorate Foundation of Northwestern Polytechnical University (cx201202 and cx201326) and the project NEXLIZ – CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

[†]E-mail address: libinlong@nwpu.edu.cn (B. Li)

[‡]E-mail address: bo.ning@tju.edu.cn (B. Ning)

[§]E-mail address: sgzhang@nwpu.edu.cn (S. Zhang).

For a given graph H , we say that G is H -free if G contains no induced copy of H . If G is H -free, then we call H a *forbidden subgraph* of G . Note that if H_1 is an induced subgraph of a graph H_2 , then an H_1 -free graph is also H_2 -free.

We first give a fundamental sufficient degree condition for hamiltonicity of graphs.

Theorem 1 (Dirac [6]). *Let G be a graph on $n \geq 3$ vertices. If every vertex of G has degree at least $n/2$, then G is hamiltonian.*

The graph $K_{1,3}$ is called the *claw*, and its only vertex of degree 3 is called its *center*. For a given graph H , we call a vertex v of H an *end-vertex* of H if $d_H(v) = 1$. Thus a claw has three end-vertices. In this paper, we use the common term claw-free graphs for $K_{1,3}$ -free graphs.

Hamiltonian properties of claw-free graphs have been well studied by many graph theorists. The lower bound on the degrees in Dirac's theorem can be lowered to roughly $n/3$ in the case of (2-connected) claw-free graphs.

Theorem 2 (Matthews and Sumner [8]). *Let G be a 2-connected claw-free graph on n vertices. If every vertex of G has degree at least $(n - 2)/3$, then G is hamiltonian.*

Forbidden subgraph conditions for hamiltonicity of graphs have also received much attention. As K_2 -free graphs are precisely the edgeless graphs, it is natural to assume that, throughout this paper, all forbidden subgraphs under consideration will have at least three vertices. We also note that every connected P_3 -free graph is a complete graph, and thus it is trivially hamiltonian if it has at least 3 vertices. It is in fact easy to show that P_3 is the only connected graph R such that every 2-connected R -free graph is hamiltonian.

Bedrossian [1] characterized all the pairs of forbidden subgraphs for hamiltonicity, excluding P_3 .

Theorem 3 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being R -free and S -free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Fig. 1).*

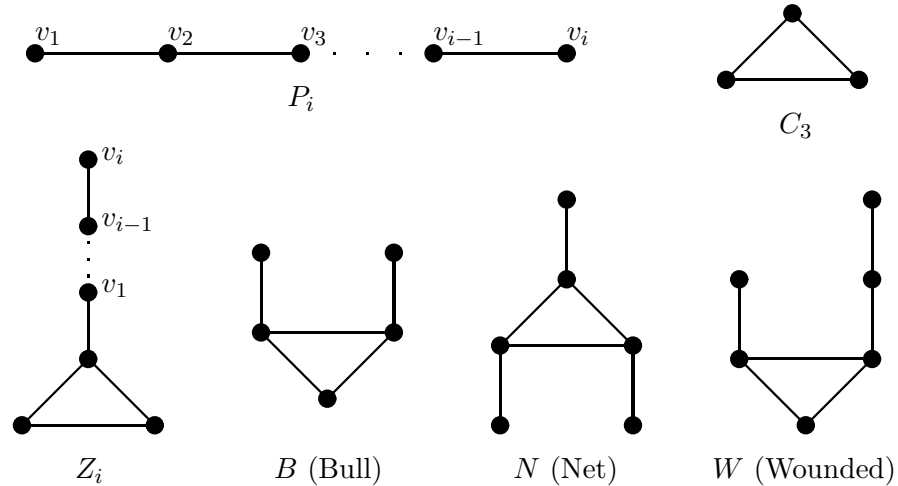


Fig. 1. Graphs P_i, C_3, Z_i, B, N and W .

Note here that the claw is always one of the forbidden subgraphs. Also recall that a P_4 -free graph is P_5 -free, etc., so the relevant graphs for S (in Theorem 3) are in fact P_6, N and W . All the other listed graphs are induced subgraphs of P_6, N or W .

At the workshop Cycles and Colourings 93 (Slovakia), Broersma [3] proposed the following conjecture.

Conjecture 1 (Broersma [3]). Let G be a 2-connected claw-free graph on n vertices. If every vertex of G which is an end-vertex of an induced copy of N in G , has degree at least $(n - 2)/3$, then G is hamiltonian.

This conjecture is still open. Fujisawa and Yamashita [7] obtained a similar result as follows.

Theorem 4 (Fujisawa and Yamashita [7]). Let G be a 2-connected claw-free graph on n vertices. If every vertex which is an end-vertex of an induced copy of Z_1 in G has degree at least $(n - 2)/3$, then G is hamiltonian.

Let G be a graph on n vertices and H a given graph. We say that G satisfies $\Phi(H, k)$ if for every vertex v which is an end-vertex of an induced copy of H in G , $d(v) \geq (n + k)/3$.

In any connected graph, a vertex which is not an end-vertex of an induced P_3 will be adjacent to all other vertices. Thus a graph satisfying $\Phi(P_3, -2)$ implies that every vertex of it has degree at least $(n - 2)/3$. By Theorem 2, such a graph is hamiltonian if it is 2-connected and claw-free. Also note that Theorem 4 implies that every 2-connected claw-free graph satisfying $\Phi(Z_1, -2)$ is hamiltonian. Motivated by Conjecture 1 and Theorem 4, we consider in this paper, the following question: For which graphs H , every 2-connected claw-free graph satisfying $\Phi(H, -2)$ is hamiltonian?

First, for a given connected graph H , note that if a graph is H -free, then it naturally satisfies $\Phi(H, -2)$. To guarantee a 2-connected claw-free graph satisfying $\Phi(H, -2)$ is hamiltonian, by Theorem 3, we can get that H must be one of the graphs in $\{P_3, P_4, P_5, P_6, C_3, Z_1, Z_2, B, N, W\}$ (to avoid the discussion of trivial cases, we assume that H has at least three vertices). Note that C_3 has no end-vertex, and every graph satisfies $\Phi(C_3, -2)$ naturally. Since not every 2-connected claw-free graph is hamiltonian, C_3 does not meet our result. Another counterexample is Z_2 . The graph in Fig. 2 is 2-connected claw-free and satisfies $\Phi(Z_2, -2)$ but it is not hamiltonian. Thus we have the following result.

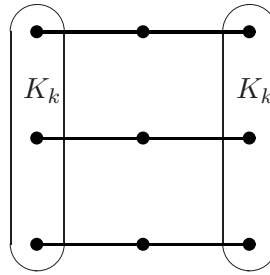


Fig. 2. A graph satisfying $\Phi(Z_2, -2)$.

Proposition 1. Let H be a connected graph on at least 3 vertices and let G be a 2-connected claw-free graph. If G satisfying $\Phi(H, -2)$ implies G is hamiltonian, then $H = P_3, P_4, P_5, P_6, Z_1, B, N$ or W .

What about the converse? Is every 2-connected claw-free graph satisfying $\Phi(H, -2)$ hamiltonian for all the graphs H listed in Proposition 1?

Note that if a graph G satisfies $\Phi(P_i, k)$, then it also satisfies $\Phi(P_j, k)$ for $j \geq i$. Also note that if G satisfies $\Phi(Z_1, k)$, then it also satisfies $\Phi(B, k)$; and if G satisfies $\Phi(B, k)$,

then it also satisfies $\Phi(N, k)$. (We remark that a graph satisfying $\Phi(Z_2, k)$ cannot ensure it satisfies $\Phi(W, k)$, although Z_2 is an induced subgraph of W .) So, in the following, we just consider the three graphs P_6 , N and W . We propose the following problem:

Problem 1. Let $H = P_6$, N or W . Is every 2-connected claw-free graph satisfying $\Phi(H, -2)$ hamiltonian?

We believe that the answer to Problem 1 is positive, but the proof may need more technical discussions. However, we can prove a slightly weaker result as follows.

Theorem 5. Let $H = P_6$, N or W , and let G be a 2-connected claw-free graph. If G satisfies $\Phi(H, 3)$, then G is hamiltonian.

Note that the graph in Fig. 2 satisfies $\Phi(Z_2, 3)$ when $k \geq 6$. Combining with Proposition 1 and Theorem 5 yields our main theorem.

Theorem 6. Let H be a connected graph on at least 3 vertices and let G be a 2-connected claw-free graph. Then G satisfying $\Phi(H, 3)$ implies G is hamiltonian, if and only if $H = P_3, P_4, P_5, P_6, Z_1, B, N$ or W .

Note that the case of $H = N$ in Theorem 6 shows that every 2-connected claw-free graph G is hamiltonian if every vertex of G which is an end-vertex of an induced copy of N , has degree at least $|V(G)|/3 + 1$. This gives an affirmative solution of the conjecture of Broersma up to an additive constant.

2 Some preliminaries

Two famous conjectures in the field of hamiltonicity of graphs are Thomassen's conjecture [10] that every 4-connected line graph is hamiltonian and Matthews and Sumner's conjecture [8] that every 4-connected claw-free graph is hamiltonian. Ryjáček proved these two conjectures are equivalent. One major tool for the proof is his closure theory [9]. Now we introduce Ryjáček's closure theory, which we will use in our proof.

Let G be a claw-free graph and x a vertex of G . Following the terminology of Ryjáček [9], we call x an *eligible* vertex if $N(x)$ induces a connected graph but is not a clique in G . The *completion* of G at x , denoted by G'_x , is the graph obtained from G by adding all missing edges uv with $u, v \in N(x)$.

Note that if a vertex, say v , has a complete neighborhood in G , i.e., $G[N(v)]$ is complete, then it also has a complete neighborhood in G'_x ; also note that if P' is an induced path in G'_x , then there is an induced path P in G with the same end-vertices such that $V(P) \subset V(P') \cup \{x\}$.

Let G be a claw-free graph. The *closure* of G , denoted by $cl(G)$, is the graph defined by a sequence of graphs G_1, G_2, \dots, G_t , and vertices x_1, x_2, \dots, x_{t-1} such that

- (1) $G_1 = G$, $G_t = cl(G)$;
- (2) x_i is an eligible vertex of G_i , $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and
- (3) G_t has no eligible vertices.

By $c(G)$ we denote the length of a longest cycle of G .

Theorem 7 (Ryjáček [9]). Let G be a claw-free graph. Then

- (1) the closure $cl(G)$ is well-defined;

- (2) there is a triangle-free graph H such that $cl(G)$ is the line graph of H ; and
- (3) $c(G) = c(cl(G))$.

Clearly every vertex has degree in $cl(G)$ not less than that in G . Ryjáček proved that if G is claw-free, then so is $cl(G)$. A claw-free graph is said to be *closed* if it has no eligible vertices. The following properties of a closed claw-free graph are obvious, and we omit the proofs.

Lemma 1. *Let G be a closed claw-free graph. Then*

- (1) every vertex is contained in exactly one or two maximal cliques;
- (2) two distinct maximal cliques have at most one common vertex;
- (3) if two vertices are nonadjacent, then they have at most two common neighbors; and
- (4) if a vertex has two neighbors in a maximal clique, then it is contained in the clique.

Now we introduce some new terminology which is useful for our proof. Let G be a claw-free graph and K a maximal clique of $cl(G)$. We call $G[K]$ a *region* of G . For a vertex v of G , we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices u, v of G , we say that they are *associated* if they are in a common region, and *dissociated* otherwise. We use the notations $u \sim v$ ($u \not\sim v$) to express the statement that u and v are associated (dissociated). So two vertices are associated in G if and only if they are adjacent in $cl(G)$. Now we can reformulate Lemma 1 as follows.

Lemma 2. *Let G be a claw-free graph. Then*

- (1) every vertex is either an interior vertex of a region, or a frontier vertex of two regions;
- (2) every two regions are either disjoint or have only one common vertex;
- (3) every two dissociated vertices have at most two common neighbors; and
- (4) if a vertex is associated with two vertices in a common region, then it is also contained in the region.

We can also get the following

Lemma 3. *Let G be a claw-free graph. Then*

- (1) if v is a frontier vertex of two regions R, R' , then $N_R(v), N_{R'}(v)$ are cliques;
- (2) if R is a region of G , then $cl(R)$ is complete;
- (3) if v is a frontier vertex and R is a region containing v , then v has an interior neighbor in R or R is complete and has no interior vertices; and
- (4) if $u \sim v$, then there is an induced path from u to v such that all internal vertices are interior vertices in the region containing u and v .

Proof. (1) If there are two neighbors x, x' of v in R such that $xx' \notin E(G)$, then let y be a neighbor of v in R' . Note that y is nonadjacent to x, x' ; otherwise it will be contained in R . Now the subgraph induced by $\{v, x, x', y\}$ is a claw, a contradiction. Thus $N_R(v)$, and similarly, $N_{R'}(v)$, is a clique.

(2) Let $K = V(R)$. Let G_1, G_2, \dots, G_t be the sequence of graphs, and x_1, x_2, \dots, x_{t-1} the sequence of vertices in the definition of $cl(G)$. Note that for every $i \leq t-1$, x_i has a complete neighborhood in G_{i+1} , and then in $cl(G)$. This implies that x_i is an interior vertex. Thus if $x_i \notin K$, then the completion of G_i at x_i does not change the structure of $G_i[K]$. Let $x_{k_1}, \dots, x_{k_{t'-1}}$ be the subsequence of x_1, \dots, x_{t-1} containing all

vertices $x_{k_i} \in K$. Note that $N_{G_{k_i}}(x_{k_i}) \subset K$. Thus x_{k_i} is an eligible vertex of $G_{k_i}[K]$ and $(G_{k_i}[K])'_{x_{k_i}} = G_{k_i+1}[K]$. Thus we have that $\text{cl}(R) = \text{cl}(G)[K]$ is the complete subgraph of $\text{cl}(G)$ corresponding to R .

(3) If R is complete in G , then either v has an interior neighbor in R or R has no interior vertices. Now we assume that R is not complete. By (2), $\text{cl}(R) = \text{cl}(G)[V(R)]$ is complete. This implies that R has at least one eligible vertex, and then, R has at least one interior vertex. If v is nonadjacent to any interior vertex in R , then the completion of an eligible vertex in R does not change the neighborhood of v . Thus v will have no interior neighbors in R in the closure $\text{cl}(R)$, a contradiction to that $\text{cl}(R)$ is a clique.

(4) Let R be the region of G containing u and v . We use the notation in the proof of (2). Note that for an induced path P' in $G_{k_i+1}[V(R)]$ connecting u and v , there is also an induced path P in $G_{k_i}[V(R)]$ connecting u and v such that $V(P) \subset V(P') \cup \{x_{k_i}\}$. This implies that there is an induced path P in R connecting u and v such that $V(P) \subset \{u, v\} \cup \{x_{k_i} : 1 \leq i \leq t' - 1\}$. Note that every x_{k_i} is an interior vertex of R . The proof is complete. \square

In the case that $u \sim v$, we use $\Pi[uv]$ to denote an induced path from u to v such that all internal vertices are interior vertices in the region containing u and v . For an induced path $P = v_0v_1v_2 \cdots v_k$ in $\text{cl}(G)$, we denote $\Pi[P] = \Pi[v_0v_1]v_1\Pi[v_1v_2]v_2 \cdots v_{k-1}\Pi[v_{k-1}v_k]$ (note that $\Pi[P]$ is an induced path of G).

Following [4], we denote by \mathcal{P} the class of all graphs that are obtained by taking two disjoint triangles $a_1a_2a_3a_1$, $b_1b_2b_3b_1$, and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} = a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$ for $k_i \geq 3$ or by a triangle $a_i b_i c_i a_i$. We denote a graph from \mathcal{P} by P_{x_1, x_2, x_3} , where $x_i = k_i$ if a_i, b_i are joined by a path P_{k_i} , and $x_i = T$ if a_i, b_i are joined by a triangle.

Theorem 8 (Brousek [4]). *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph in \mathcal{P} .*

We mention the following result deduced from Brousek et al. [5] to complete this section.

Theorem 9 (Brousek et al. [5]). *Let G be a claw-free graph. If G is N -free, then $\text{cl}(G)$ is also N -free.*

3 Proof of Theorem 5

Assume that G is not hamiltonian. By Theorems 7 and 8, $\text{cl}(G)$ contains an induced subgraph $P_{x_1, x_2, x_3} \in \mathcal{P}$. We use the notation a_i, b_i, c_i and c_i^j defined in Section 2. If $x_i = k_i$, then let P^i be the path $a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$; if $x_i = T$, then let $P^i = a_i b_i$. Let A be the region of G containing the vertices a_1, a_2, a_3 , B be the region of G containing the vertices b_1, b_2, b_3 . Note that A and B are possibly not disjoint. If they are not disjoint, then let c be the only common vertex of A and B . Clearly, a_i, b_i and c (if exists) are all frontier vertices. If $x_i = T$, then let a'_i be the successor of a_i in $\Pi[a_i c_i]$ and b'_i be the successor of b_i in $\Pi[b_i c_i]$; if $x_i = k_i$, then let a'_i be the successor of a_i in $\Pi[a_i c_i^1]$ and b'_i be the successor of b_i in $\Pi[b_i c_i^{k_i-2}]$.

In this section, we say that a vertex is *hefty* if it has degree at least $n/3 + 1$.

Claim 1. Let v_1, v_2, v_3 be three pairwise nonadjacent vertices of G .

- (1) If $v_1 \not\sim v_2$, $v_1 \not\sim v_3$ and v_2, v_3 have at most one common neighbor, then one of v_1, v_2, v_3 is not hefty.
- (2) If $v_1 \not\sim v_2$, $v_1 \not\sim v_3$ and $v_2 \not\sim v_3$, then one of v_1, v_2, v_3 is not hefty.

Proof. (1) By Lemma 2 (3), $|N(v_1) \cap N(v_2)| \leq 2$ and $|N(v_1) \cap N(v_3)| \leq 2$. Note that $|N(v_2) \cap N(v_3)| \leq 1$. If all these three vertices are hefty, i.e., $d(v_i) \geq n/3 + 1$ for $i = 1, 2, 3$, then

$$n \geq 3 + \sum_{1 \leq i \leq 3} d(v_i) - \sum_{1 \leq i < j \leq 3} |N(v_i) \cap N(v_j)| \geq 3 + 3 \left(\frac{n}{3} + 1 \right) - 5 = n + 1,$$

a contradiction.

(2) By (1) and Lemma 2 (3), each of $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$ has exactly two common neighbors. Let u_{ij} and u'_{ij} be the two common neighbors of v_i and v_j . By Lemma 2 (4), $u_{ij} \not\sim u'_{ij}$. This implies that all the three vertices v_1, v_2, v_3 are frontier vertices. Moreover, by applying a similar argument as in (1), we have

$$n \geq 3 + d(v_1) + d(v_2) + d(v_3) - 6 \geq 3 \cdot \left(\frac{n}{3} + 1 \right) - 3 = n.$$

This implies that every vertex of G is adjacent to at least one vertex in $\{v_1, v_2, v_3\}$. Thus G consists of the six regions containing v_1, v_2 and v_3 , and all the six regions are cliques by Lemma 3 (1).

Since $u_{12} \not\sim u'_{12}$ and $u_{13} \not\sim u'_{13}$ and all the four vertices are adjacent to v_1 , we have either $u_{12} \sim u_{13}$ or $u_{12} \sim u'_{13}$. We assume without loss of generality that $u_{12} \sim u_{13}$, which implies that $u_{12}u_{13} \in E(G)$. Now we can begin with the cycle $C_0 = v_1u'_{12}v_2u_{12}u_{13}v_3u'_{13}v_1$, and add other vertices, one by one, to the cycle at the place between two associated vertices, and finally obtain a Hamilton cycle of G , a contradiction. \square \square

The case $H = P_6$

Let $P = a'_1a_1II[a_1a_2]a_2II[P^2]b_2II[b_2b_3]b_3b'_3$. Note that P is an induced copy of P_l with $l \geq 6$. This implies that a'_1 , and similarly, a'_2, a'_3 , are hefty. Note that a'_1, a'_2 and a'_3 are pairwise dissociated in G , a contradiction to Claim 1.

The case $H = N$

Claim 2. There are at least two hefty vertices in A (and similarly, in B).

Proof. Let $G' = G[V(A) \cup \{a'_1, a'_2, a'_3\}]$. From Lemma 3 (2), we can see that $\text{cl}(G') = \text{cl}(G)[V(G')]$. Note that the subgraph of $\text{cl}(G)[V(G')]$ induced by $\{a_1, a'_1, a_2, a'_2, a_3, a'_3\}$ is an N . By Theorem 9, G' contains an induced N . This implies that $V(G')$ contains at least three pairwise nonadjacent hefty vertices. If two of them are not in A , then we assume without loss of generality that a'_1, a'_2 are hefty. Note that the third hefty vertex is in $(V(A) \cup \{a'_3\}) \setminus \{a_1, a_2\}$. This implies that the three hefty vertices are pairwise dissociated, a contradiction to Claim 1. \square \square

Let b, b' be two hefty vertices in B . Set

$$N_i = \{v \in V(A) : d_A(a_1, v) = i\} \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{a_1\}$ and $N_1 = N_A(a_1)$. In addition, we define that $N_{-1} = \{a'_1\}$. Note that for any vertex $v \in N_i$, with $1 \leq i \leq j$, v has a neighbor in N_{i-1} . Also note that if

v has a neighbor in N_{i+1} , $1 \leq i \leq j-1$, then by Lemma 3 (1), v is an interior vertex, especially, v is not a_2, a_3 and c .

Claim 3. N_i is a clique for all $1 \leq i \leq j$.

Proof. We use induction on i . By Lemma 3 (1), N_1 is a clique. Now we assume that $2 \leq i \leq j$. Note that N_{i-1}, N_{i-2} and N_{i-3} are nonempty.

Assume that there are two vertices y, y' in N_i with $yy' \notin E(G)$. If y and y' have a common neighbor in N_{i-1} , then let x be a common neighbor of y and y' in N_{i-1} , and w be a neighbor of x in N_{i-2} . Then the subgraph induced by $\{x, w, y, y'\}$ is a claw, a contradiction. This implies that y and y' have no common neighbors in N_{i-1} . Now let x be a neighbor of y in N_{i-1} and x' be a neighbor of y' in N_{i-1} . Note that $xy', x'y \notin E(G)$. Let w be a neighbor of x in N_{i-2} and let v be a neighbor of w in N_{i-3} . By the induction hypothesis, $xx' \in E(G)$. If $wx' \notin E(G)$, then the subgraph induced by $\{x, w, x', y\}$ is a claw, a contradiction. This implies that $wx' \in E(G)$. Now the subgraph induced by $\{w, v, x, y, x', y'\}$ is an N . Thus the three vertices v, y and y' are all hefty.

By Lemma 2 (4), $v \not\sim b$ or $v \not\sim b'$. We assume without loss of generality that $v \not\sim b$. Similarly $b \not\sim y$ or $b \not\sim y'$, we assume without loss of generality that $b \not\sim y$. Note that b, v, y are all hefty, $b \not\sim v$, $b \not\sim y$ and v, y have no common neighbors. We get a contradiction. \square

If both a_2 and a_3 are in N_j , then let w be a neighbor of a_2 in N_{j-1} , v be a neighbor of w in N_{j-2} . By Claim 3 and Lemma 3 (1), $a_2a_3, wa_3 \in E(G)$. Thus the subgraph induced by $\{w, v, a_2, a'_2, a_3, a'_3\}$ is an N . Thus v, a'_2 and a'_3 are three hefty vertices. Note that v, a'_2 and a'_3 are pairwise dissociated, a contradiction. So we assume without loss of generality that $a_2 \notin N_j$.

Let $a_2 \in N_i$, where $1 \leq i \leq j-1$. Let y be a vertex in N_{i+1} . Recall that a_2 has no neighbors in N_{i+1} . Let x be a neighbor of y in N_i , w be a neighbor of a_2 in N_{i-1} and v be a neighbor of w in N_{i-2} . By Claim 3 and Lemma 3 (1), $a_2x, wx \in E(G)$, and the subgraph induced by $\{w, v, x, y, a_2, a'_2\}$ is an N . Thus v, y and a'_2 are three hefty vertices. Note that $a'_2 \not\sim v$, $a'_2 \not\sim y$, and v, y have no common neighbors, a contradiction.

The case $H = W$

Claim 4. For i, j , $1 \leq i < j \leq 3$, one of the edges in $\{a_ia_j, b_ib_j, a_ib_i, a_jb_j\}$ is not in $E(G)$.

Proof. We assume that $a_ia_j, b_ib_j, a_ib_i, a_jb_j \in E(G)$. By Lemma 3 (1), $a'_ib_i, a'_jb_j \in E(G)$. Let a be the successor of a_j in the path $\Pi[a_ka_k]$, where $k \neq i, j$. Then the subgraph induced by $\{a'_j, a_j, a, b_j, b_i, a'_i\}$ is a W . Thus a, a'_i , and similarly a'_j , are hefty. Note that a, a'_i and a'_j are pairwise dissociated, a contradiction. \square

As in the case of N , we set

$$N_i = \{v \in V(A) : d_A(a_1, v) = i\} \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{a_1\}$, $N_1 = N_A(a_1)$ and we define additionally $N_{-1} = \{a'_1\}$.

Claim 5. There is a hefty vertex in $A \setminus \{a_1, a_2, a_3, c\}$ (and similarly, in $B \setminus \{b_1, b_2, b_3, c\}$).

Proof. We assume on the contrary that there are no hefty vertices in $A \setminus \{a_1, a_2, a_3, c\}$.

Claim 5.1. N_i is a clique for all $1 \leq i \leq j$.

Proof. We use induction on i . By Lemma 3 (1), N_1 is a clique. Now we assume that $2 \leq i \leq j$. Note that N_{i-1}, N_{i-2} and N_{i-3} are nonempty.

Assume that there are two vertices y, y' in N_i with $yy' \notin E(G)$. Note that y and y' have no common neighbors in N_{i-1} . Let x be a neighbor of y in N_{i-1} , x' be a neighbor of y' in N_{i-1} , w be a neighbor of x in N_{i-2} and v be a neighbor of w in N_{i-3} . By the induction hypothesis, $xx' \in E(G)$. Note that $wx' \in E(G)$; otherwise the subgraph induced by $\{x, w, x', y\}$ is a claw.

If $y = a_2$, then the subgraph induced by $\{x', w, v, x, a_2, a'_2\}$ and the subgraph induced by $\{w, x', y', x, a_2, a'_2\}$ are W 's. Thus v, y' and a'_2 are three hefty vertices. Note that $a'_2 \not\sim v$, $a'_2 \not\sim y'$, and v, y' have no common neighbors, a contradiction. So we assume that $y \neq a_2$, and similarly, $y \neq a_3$, $y' \neq a_2$, $y' \neq a_3$. This implies that either y or y' is in $A \setminus \{a_1, a_2, a_3, c\}$.

We assume without loss of generality that $y \in A \setminus \{a_1, a_2, a_3, c\}$. Let P' be a shortest path from w to a_1 (note that P' consists of the vertex a_1 if $w = a_1$). Let w, v and u be the first three vertices in the path $P = P'a_1\Pi[P^1]b_1\Pi[b_1b_2]$. Then the subgraph induced by $\{x', x, y, w, v, u\}$ is a W . Thus y is a hefty vertex, a contradiction. \square \square

If both a_2 and a_3 are in N_j , then let w be a neighbor of a_2 in N_{j-1} , v be a neighbor of w in N_{j-2} . By Claim 5.1 and Lemma 3 (1), $a_2a_3, wa_3 \in E(G)$. Let a_2, y and z be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. By Claim 4, $a_3z \notin E(G)$. Then the subgraph induced by $\{a_3, w, v, a_2, y, z\}$ is a W . Let a_3, y', z' be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_1]$. By Claim 4, $wz' \notin E(G)$. Then the subgraph induced by $\{w, a_2, a'_2, a_3, y', z'\}$ is a W . Thus v, a'_2 , and similarly, a'_3 , are hefty. Note that v, a'_2 and a'_3 are pairwise dissociated, a contradiction. So we assume without loss of generality that $a_2 \notin N_j$.

Let $a_2 \in N_i$, where $1 \leq i \leq j-1$. Let y be a vertex in N_{i+1} . Recall that a_2 has no neighbors in N_{i+1} . Let x be a neighbor of y in N_i , w be a neighbor of a_2 in N_{i-1} and v be a neighbor of w in N_{i-2} . Note that $a_2x, wx \in E(G)$.

If $y = a_3$, then let $z = a'_3$; and if $y = c$, then let z be the successor of c in $\Pi[cb_3]$. Then the subgraph induced by $\{a_2, w, v, x, y, z\}$ and the subgraph induced by $\{w, a_2, a'_2, x, y, z\}$ are W 's. Thus v, a'_2 and z are hefty. Note that v, a'_2 and z are pairwise dissociated, a contradiction. Now we assume that $y \neq c, a_3$. Let a_2, y', z' be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. Then the subgraph induced by $\{w, x, y, a_2, y', z'\}$ is a W . This implies that y is hefty, a contradiction. \square \square

Now let a and b be two hefty vertices in $A \setminus \{a_1, a_2, a_3, c\}$ and $B \setminus \{b_1, b_2, b_3, c\}$, respectively. Since a, b and a'_i are pairwise dissociated, a'_i is not hefty.

By Lemma 3 (3), a_1 has an interior neighbor in A or $a_1a \in E(G)$. In any case, a_1 has a neighbor in $A \setminus \{a_2, a_3, c\}$. If $a_1a_2 \in E(G)$, then let v be a neighbor of a_1 in $A \setminus \{a_2, a_3, c\}$. By Lemma 3, $a_2v \in E(G)$. Let a_2, x and y be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. Then the subgraph induced by $\{v, a_1, a'_1, a_2, x, y\}$ is a W . Thus a'_1 is hefty, a contradiction. This implies that a_1a_2 , and similarly, a_1a_3, a_2a_3 , is not in $E(G)$.

Claim 6. N_i is a clique for all $1 \leq i \leq j$.

Proof. We use induction on i . By Lemma 3 (1), N_1 is a clique.

We first consider the case $i = 2$. Recall that $a_1a_2 \notin E(G)$, which implies that $a_2 \notin N_1$. If $a_2 \in N_2$, then let $z = a'_2, y = a_2$; and if $a_2 \notin N_2$, then ($j \geq 3$ and) let z be a vertex in N_3 , and y be a neighbor of z in N_2 .

We claim that y is adjacent to every vertex in $N_2 \setminus \{y\}$. Assume that $yy' \notin E(G)$ for $y' \in N_2 \setminus \{y\}$. Then y and y' have no common neighbors in N_1 . Let x be a neighbor of y in N_1 and x' be a neighbor of y' in N_1 . Then $xy', x'y \notin E(G)$. Since $xx' \in E(G)$, the subgraph induced by $\{x', a_1, a'_1, x, y, z\}$ is a W , and this implies that a'_1 is hefty, a contradiction. Thus, as we claimed, y is adjacent to every vertex in $N_2 \setminus \{y\}$. Now let y', y'' be two vertices in $N_2 \setminus \{y\}$. We claim that $y'y'' \in E(G)$. If $y'z \in E(G)$, then ($z \neq a'_2$ and) similarly as the case of y , we can see that y' is adjacent to every vertex in $N_2 \setminus \{y'\}$, including y'' . So we assume that $y'z$, and similarly, $y''z$, is not in $E(G)$. Then the subgraph induced by $\{y, y', y'', z\}$ is a claw, a contradiction. Thus, as we claimed, N_2 is a clique.

Now we assume that $3 \leq i \leq j$. Note that $N_{i-1}, N_{i-2}, N_{i-3}$ and N_{i-4} are nonempty.

Assume that there are two vertices z and z' in N_i with $zz' \notin E(G)$. Note that z and z' have no common neighbors in N_{i-1} . Let y be a neighbor of z in N_{i-1} and y' be a neighbor of z' in N_{i-1} . Then $yz', y'z \notin E(G)$. Let x be a neighbor of y in N_{i-2} , w be a neighbor of x in N_{i-3} and v be a neighbor of w in N_{i-4} . Then $yy', xy' \in E(G)$. Now the subgraph induced by $\{y', y, z, x, w, v\}$ is a W . Thus v and z are hefty. Note that $b \not\sim v, b \not\sim z$, and v, z have no common neighbors, a contradiction. \square

Recall that $a_2a_3 \notin E(G)$, which implies that either a_2 or $a_3 \notin N_j$. Also recall that $a_2, a_3 \notin N_1$. We assume without loss of generality that $a_2 \in N_i$, where $2 \leq i \leq j-1$. Let z be a vertex in N_{i+1} , y be a neighbor of z in N_i , x be a neighbor of a_2 in N_{i-1} , w be a neighbor of x in N_{i-2} and v be a neighbor of w in N_{i-3} . By Claim 6 and Lemma 3 (1), $a_2y, xy \in E(G)$. Then the subgraph induced by $\{y, a_2, a'_2, x, w, v\}$ is a W . This implies that a'_2 is hefty, a contradiction.

The proof is complete. \square

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